Fractional order differentiation by integration with Jacobi polynomials

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Abstract—**The differentiation by integration method with Jacobi polynomials was originally introduced by Mboup, Join and Fliess [22], [23]. This paper generalizes this method from the integer order to the fractional order for estimating the fractional order derivatives of noisy signals. The proposed fractional order differentiator is deduced from the Jacobi orthogonal polynomial filter and the Riemann-Liouville fractional order derivative definition. Exact and simple formula for this differentiator is given where an integral formula involving Jacobi polynomials and the noisy signal is used without complex mathematical deduction. Hence, it can be used both for continuous-time and discrete-time models. The comparison between our differentiator and the recently introduced digital fractional order Savitzky-Golay differentiator is given in numerical simulations so as to show its accuracy and robustness with respect to corrupting noises.**

I. INTRODUCTION

Fractional models arise in many practical situations ([5], [33] for example). Such fractional order systems may also be used for control purposes: CRONE control is known to have good robustness properties (see [26], [27], [28], [33]). In order to implement such controller one needs to have a good digital fractional order differentiator from noisy signals, which is the scope of this paper.

The fractional derivative has a long history and are now very useful in science, engineering and finance (see, e.g., [25], [11], [12]). The fractional order differentiator is concerned with estimating the fractional order derivatives of an unknown signal from its noisy observed data. Because of its importance, various methods have been developed during the last years. They are divided into two kinds: continuous-time model (see, e.g., [32], [2], [29]) and discrete-time model (see, e.g., [38], [35], [39]). Nevertheless, the real case of signals with noises was somewhat overlooked. In order to resolve this problem, an optimization formulation using genetic algorithms was proposed in [20] to reduce the noise effect in the estimations of the fractional order derivatives. But, the complex mathematical deduction restricts its application. A novel **D**igital **F**ractional **O**rder **S**avitzky-**G**olay **D**ifferentiator (DFOSGD) was introduced in [3], which was deduced from the Riemann-Liouville fractional order derivative definition [31] (p. 63) and the Savitzky-Golay filter [36], [37]. Using the Savitzky-Golay filter guarantees its accuracy and simplicity for estimating the fractional order derivatives of noisy signals. However, let us recall that the Legendre polynomial filter [30] is more recommended than the Savitzky-Golay filter for reasons of simplicity and speed. In particular, it has advantages of being suitable for irregularly spaced or missing data.

The method of *differentiation by integration* uses an integral of an unknown noisy signal to estimate the integer order derivatives of this signal. For free of noise signals, this method was firstly studied by Cioranescu [4] (1938) and became well known for the Lanczos generalized derivative [13] (p. 324) (1956). The Lanczos generalized derivative proposed an integral of a noisy signal and the Legendre orthogonal polynomial to estimate the first order derivative of the signal. Recently, in the noisy case, Mboup, Join and Fliess [22], [23] applied an algebraic setting to estimate high order derivatives by integration, where Jacobi orthogonal polynomials were introduced in the integral. Hence, we call the obtained differentiator *Jacobi differentiator*. Moreover, it was shown in [15] that the Jacobi differentiator greatly improved the convergence rate of the Lanczos generalized derivative. Very recently, it was shown in [17] that the Jacobi differentiator could also be obtained by taking the derivative of the Jacobi orthogonal polynomial filter considered as the extension of the Legendre polynomial filter [30], [24].

Let us recall that the algebraic differentiation method used in [22], [23] was introduced in [9] and also analyzed in [14], [15], [16], [17], [34], where the used algebraic manipulations were inspired by the algebraic parametric estimation methods [10], [21], [18], [19]. Additional theoretical foundations can be found in [6], [8]. In particular, by using non standard analysis techniques, Fliess [6], [7] showed that these methods exhibit good robustness properties with respect to corrupting noises without the need of knowing their statistical properties. However, these methods have not been used to estimate fractional order derivatives.

The aim of this paper is to generalize the differentiation by integration method from the integer order to the fractional order for estimating fractional order derivatives. In Section II, we recall the method of integer order differentiation by integration with Jacobi polynomials. Then, we deduce a fractional order differentiator from the Riemann-Liouville fractional order derivative definition and the Jacobi orthogonal polynomial filter. This differentiator is exactly given by an

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integral of Jacobi polynomials. Hence, we call it *fractional Jacobi differentiator*. In Section III, we compare the fractional Jacobi differentiator to the DFOSGD in some numerical simulations. Finally, we give some conclusions and perspectives for our future work in Section IV.

II. METHODOLOGY

Let $y^{\delta} = y + \varpi$ be a noisy signal observed in an interval $I = [a, b] \subset \mathbb{R}$ of length $h = b - a$, where $y \in \mathcal{C}(I)$ and the noise[1] ϖ is bounded and integrable. We are going to estimate the α^{th} ($\alpha \in \mathbb{R}_+$) order derivative of y by using its observation y^{δ}.

A. Differentiation by integration with Jacobi polynomials

In this subsection, we recall the method of integer order differentiation by integration with Jacobi polynomials. This method was introduced in [22], [23] and studied in [14], [15], [16], [17], [34].

The n^{th} ($n \in \mathbb{N}$) order Jacobi orthogonal polynomial defined on $[0,1]$ is given as follows (see [1] p. 775)

$$P_n^{(\mu,\kappa)}(t) = \sum_{j=0}^{n} \binom{n+\mu}{j} \binom{n+\kappa}{n-j} (t-1)^{n-j}\, t^j, \tag{1}$$

where $\mu, \kappa \in]-1, +\infty[$. Let f and g be two functions belonging to $\mathcal{C}([0,1])$, then we define the scalar product $\langle \cdot, \cdot \rangle_{\mu,\kappa}$ of these functions by (see [1] p. 774)

$$\langle f(\cdot), g(\cdot) \rangle_{\mu,\kappa} = \int_0^1 w_{\mu,\kappa}(t)\, f(t)\, g(t)\, dt, \tag{2}$$

where $w_{\mu,\kappa}(t) = (1-t)^{\mu} t^{\kappa}$ is the associated weighted function. Thus, by denoting its associated norm by $\|\cdot\|_{\mu,\kappa}$, we obtain

$$\begin{aligned} &\|P_n^{(\mu,\kappa)}\|^2_{\mu,\kappa} = \\ &\qquad \frac{\Gamma(\mu+n+1)\,\Gamma(\kappa+n+1)}{\Gamma(\mu+\kappa+n+1)\,\Gamma(n+1)\,(2n+\mu+\kappa+1)}. \end{aligned} \tag{3}$$

where $\Gamma(\cdot)$ is the classical Gamma function (see [1] p. 255).

Let us ignore the noise for a moment. Then, we define an N^{th} ($N \in \mathbb{N}$) order approximation polynomial of y by taking its truncated Jacobi orthogonal series expansion

$$\begin{aligned} &\forall\, t \in [0,1], \;\; D^{(0)}_{h,\mu,\kappa,N} y(a+ht) := \\ &\qquad \sum_{i=0}^{N} \frac{\left\langle P_i^{(\mu,\kappa)}(\cdot), y(a+h\cdot) \right\rangle_{\mu,\kappa}}{\|P_i^{(\mu,\kappa)}\|^2_{\mu,\kappa}}\; P_i^{(\mu,\kappa)}(t). \end{aligned} \tag{4}$$

If we take $\kappa = \mu = 0$, then the Jacobi orthogonal polynomials become the Legendre orthogonal polynomials. Hence, this Jacobi polynomial filter [24] can be considered as the extension of the Legendre polynomial filter [30].

[1] More generally, the noise is a stochastic process, which is bounded with certain probability and integrable in the sense of convergence in mean square (see [16]).

If $y \in \mathcal{C}^n(I)$ with $n \in \mathbb{N}$, then we take the n^{th} order derivative of the polynomial $D^{(0)}_{h,\mu,\kappa,N} y(a+h\cdot)$ as an estimate of the n^{th} order derivative of y [17]: $\forall\, t \in [0,1]$,

$$\begin{aligned} &D^{(n)}_{h,\mu,\kappa,N} y(a+ht) \\ &\quad := \frac{d^n}{d(ht)^n} \left\{ D^{(0)}_{h,\mu,\kappa,N} y(a+ht) \right\} \\ &\quad = \frac{1}{h^n} \frac{d^n}{dt^n} \left\{ D^{(0)}_{h,\mu,\kappa,N} y(a+ht) \right\}. \end{aligned} \tag{5}$$

For any $t \in [0,1]$, this differentiator can be expressed as follows [17]

$$\begin{aligned} &D^{(n)}_{h,\mu,\kappa,N} y(a+ht) = \\ &\qquad \frac{1}{h^n} \int_0^1 Q_{\mu,\kappa,n,N}(\tau,t)\, y(a+h\tau)\, d\tau, \end{aligned} \tag{6}$$

with $C_{\mu,\kappa,n,i} = \frac{(\mu+\kappa+2n+2i+1)\,\Gamma(\mu+\kappa+2n+i+1)\,\Gamma(n+i+1)}{\Gamma(\kappa+n+i+1)\,\Gamma(\mu+n+i+1)}$,

$$Q_{\mu,\kappa,n,N}(\tau,t) = w_{\mu,\kappa}(\tau) \sum_{i=0}^{N-n} C_{\mu,\kappa,n,i} P_i^{(\mu+n,\kappa+n)}(t) P_{n+i}^{(\mu,\kappa)}(\tau).$$

Finally, we replace y in (6) by y^{δ}. Consequently, the n^{th} order derivative of y can be estimated by an integral of Jacobi polynomials. The noise effect on obtained estimations was analyzed in [14], [16], [17], [23], [21]. In the next subsection, we are going to generalize this differentiation by integration method from the integer order to the fractional order.

B. Fractional order differentiation by integration

Similarly to [3], we are going to take the Riemann-Liouville fractional order derivative of our approximation polynomial so as to get a fractional order differentiator. The Riemann-Liouville fractional order derivative (see [31] p. 62) is defined as follows: $\forall\, x \in \mathbb{R}_+^*$,

$${}_0D_x^{\alpha} f(x) := \frac{1}{\Gamma(l-\alpha)} \frac{d^l}{dx^l} \int_0^x (x-\tau)^{l-\alpha-1} f(\tau)\, d\tau, \tag{7}$$

where $0 \le l-1 \le \alpha < l$ with $l \in \mathbb{N}^*$.

From now on, we denote the α^{th} ($\alpha \in \mathbb{R}_+$) order derivative of f by

$$\forall\, x \in \mathbb{R}_+^*,\;\; {}_0D_x^{\alpha} f(x) = \frac{d^{\alpha}}{dx^{\alpha}} \{f(x)\} = f^{(\alpha)}(x). \tag{8}$$

Hence, if we take $f(x) = x^n$ with $n \in \mathbb{N}$ and $x \in \mathbb{R}_+^*$, then we obtain (see [31] p. 72)

$$\frac{d^{\alpha}}{dx^{\alpha}} \{x^n\} = \frac{\Gamma(n+1)}{\Gamma(n+1-\alpha)}\, x^{n-\alpha}. \tag{9}$$

By using (9), we obtain the following lemma.

Lemma 1: The α^{th} ($\alpha \in \mathbb{R}_+$) order derivative of the n^{th} order Jacobi orthogonal polynomial $P_n^{(\mu,\kappa)}$ defined in (1) is given as follows: $\forall\, t \in]0,1]$,

$$\begin{aligned} &\frac{d^{\alpha}}{dt^{\alpha}} \left\{ P_n^{(\mu,\kappa)}(t) \right\} = \\ &\qquad \sum_{j=0}^{n} \sum_{l=0}^{n-j} c_{\mu,\kappa,n,j,l}\, \frac{\Gamma(n-l+1)}{\Gamma(n-l+1-\alpha)} t^{n-l-\alpha}, \end{aligned} \tag{10}$$

where $c_{\mu,\kappa,n,j,l} = (-1)^l \binom{n+\mu}{j}\binom{n+\kappa}{n-j}\binom{n-j}{l}$.

Proof. By applying the binomial theorem to (1), we get

$$P_n^{(\mu,\kappa)}(t) = \sum_{j=0}^{n}\sum_{l=0}^{n-j} c_{\mu,\kappa,n,j,l}\, t^{n-l},$$

where $c_{\mu,\kappa,n,j,l} = (-1)^l \binom{n+\mu}{j}\binom{n+\kappa}{n-j}\binom{n-j}{l}$. Hence, this proof can be completed by using (9) and the linearity of the fractional order differentiation (see [31] p. 91). □

Then, we give the following proposition.

Proposition 1: Let $y^\delta = y + \varpi$ be a noisy observation of y in an interval $I = [a,b] \subset \mathbb{R}$ of length $h = b - a$, where $y \in \mathcal{C}(I)$ and the noise ϖ is bounded and integrable. If the α^{th} ($\alpha \in \mathbb{R}_+$) order derivative of y exists, then a fractional order differentiator, called *fractional Jacobi differentiator*, is defined as follows: $\forall\, t \in]0,1]$

$$D^{(\alpha)}_{h,\mu,\kappa,N} y^\delta(a+ht) := \frac{1}{h^\alpha}\int_0^1 Q_{\mu,\kappa,\alpha,N}(\tau,t)\, y^\delta(a+h\tau)\, d\tau, \tag{11}$$

where $h \in \mathbb{R}_+^*$, $N \in \mathbb{N}$, $\mu,\kappa \in]-1,+\infty[$,

$$Q_{\mu,\kappa,\alpha,N}(\tau,t) = \sum_{i=0}^{N} w_{\mu,\kappa}(\tau)\frac{P_i^{(\mu,\kappa)}(\tau)}{\|P_i^{(\mu,\kappa)}\|^2_{\mu,\kappa}} q_{\mu,\kappa,i}(t), \tag{12}$$

$$q_{\mu,\kappa,i}(t) = \sum_{j=0}^{i}\sum_{l=0}^{i-j} c_{\mu,\kappa,i,j,l}\,\frac{\Gamma(i-l+1)}{\Gamma(i-l+1-\alpha)} t^{i-l-\alpha}, \tag{13}$$

with $c_{\mu,\kappa,i,j,l} = (-1)^l\binom{i+\mu}{j}\binom{i+\kappa}{i-j}\binom{i-j}{l}$ and $\|P_i^{(\mu,\kappa)}\|^2_{\mu,\kappa}$ is given in (3).

Proof. Let us take the α^{th} order derivative of the polynomial $D^{(0)}_{h,\mu,\kappa,N} y(a+h\cdot)$ defined in (4). By using the scale change property of the fractional order differentiation (see [25] p. 76) we obtain: $\forall\, t \in]0,1]$,

$$\begin{aligned} D^{(\alpha)}_{h,\mu,\kappa,N} y(a+ht) &:= \frac{d^\alpha}{d(ht)^\alpha}\left\{D^{(0)}_{h,\mu,\kappa,N} y(a+ht)\right\} \\ &= \frac{1}{h^\alpha}\frac{d^\alpha}{dt^\alpha}\left\{D^{(0)}_{h,\mu,\kappa,N} y(a+ht)\right\}. \end{aligned} \tag{14}$$

The linearity of the fractional order differentiation gives us that

$$D^{(\alpha)}_{h,\mu,\kappa,N} y(a+ht) = \frac{1}{h^\alpha}\sum_{i=0}^{N}\frac{\left\langle P_i^{(\mu,\kappa)}(\cdot), y(a+h\cdot)\right\rangle_{\mu,\kappa}}{\|P_i^{(\mu,\kappa)}\|^2_{\mu,\kappa}}\frac{d^\alpha}{dt^\alpha}\left\{P_i^{(\mu,\kappa)}(t)\right\}. \tag{15}$$

Finally, this proof can be completed by using Lemma 1 and substituting y by y^δ in (15). □

Consequently, by calculating the integral of the noisy signal y^δ and a sum of the Jacobi polynomials we can estimate the value of $y^{(\alpha)}$ at each point $a+ht$ in the interval $]a,b]$ for each $t \in]0,1]$. This integral corresponds to a convolution in the discrete case.

If we fix the value of t to 1, then the fractional Jacobi differentiator $D^{(\alpha)}_{h,\mu,\kappa,N} y^\delta(a+ht)$ only estimates the value of $y^{(\alpha)}$ at point b. Thus, if we increase the length of the interval $I = [a,b] \subset \mathbb{R}$, then we can estimate the other values of $y^{(\alpha)}$. Consequently, the fractional Jacobi differentiator can also be considered as a pointwise causal differentiator which is useful in many on-line applications. Moreover, we can fix the parameters κ, μ and N such that the sum of the Jacobi polynomials can be explicitly calculated by off-line work. Indeed, the fractional Jacobi differentiator only needs a simple convolution for on-line applications. The computation time is significatively improved.

III. SIMULATIONS RESULTS

The comparisons between the DFOSGD and some existing fractional order differentiators were shown in [3], where the DFOSGD was better than the others. In this section, by taking one numerical example considered in [3] we compare the fractional Jacobi differentiator to the DFOSGD so as to show its accuracy and robustness with respect to corrupting noises.

A. Numerical case

From now on, we assume that $y^\delta(x_i) = y(x_i) + c\varpi(x_i)$ is a sequence of uniformly sampled noisy data of y with $x_i = a + T_s i$ and $T_s = \frac{h}{M}$ for $i = 0,\cdots,M$. The noise $c\varpi(x_i)$ is assumed to a zero-mean white Gaussian iid sequence, and $c \in \mathbb{R}_+$. In this discrete case, we apply the trapezoidal numerical integration method in the fractional Jacobi differentiator to approximate the integral. Then, let us recall that the values of $y^{(\alpha)}(x_i)$ for $i = 0,\cdots,M$ are estimated by the DFOSGD as follows

$$\widetilde{y^{(\alpha)}(x_i)} = \frac{1}{(T_s)^\alpha} b_{i,N}\left(X^T_{\theta,N}X_{\theta,N}\right)^{-1} X^T_{\theta,N} Y^\delta_\theta, \tag{16}$$

where $N \in \mathbb{N}$, $\theta \in \mathbb{N}$,

$$b_{i,N} = \begin{pmatrix} \frac{1}{\Gamma(1-\alpha)}(i+1)^{-\alpha} \\ \frac{1}{\Gamma(2-\alpha)}(i+1)^{1-\alpha} \\ \frac{\Gamma(3)}{\Gamma(3-\alpha)}(i+1)^{2-\alpha} \\ \vdots \\ \frac{\Gamma(N+1)}{\Gamma(N+1-\alpha)}(i+1)^{N-\alpha} \end{pmatrix}^T, \quad Y^\delta_\theta = \begin{pmatrix} y^\delta(x_0) \\ y^\delta(x_\theta) \\ y^\delta(x_{2\theta}) \\ \vdots \\ y^\delta(x_M) \end{pmatrix}, \tag{17}$$

and

$$X_{\theta,N} = \begin{pmatrix} 1 & 1^1 & \cdots & 1^N \\ 1 & (1+\theta)^1 & \cdots & (1+\theta)^N \\ 1 & (1+2\theta)^1 & \cdots & (1+2\theta)^N \\ \vdots & \vdots & \vdots & \vdots \\ 1 & (M+1)^1 & \cdots & (M+1)^N \end{pmatrix}. \tag{18}$$

Hence, the idea is to obtain an N^{th} order approximation polynomial by using Savitzky-Golay algorithm [36], [37] from a subsequence Y^δ_θ, and then to calculate the α^{th} order derivative of this polynomial at each point x_i.

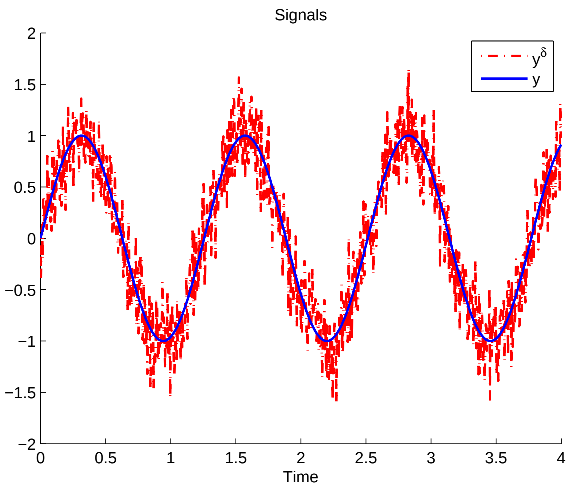

Fig. 1. Signal y and noisy signal y^{δ}.

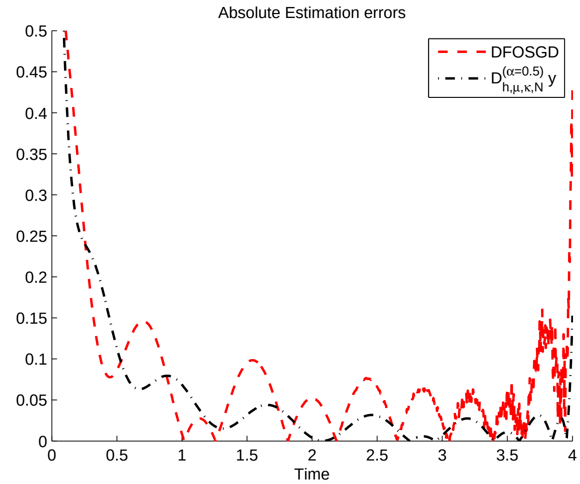

Fig. 2. Absolut estimation errors in noise-free case.

B. Example

In this example, we assume that $y^{\delta}(x_i) = \sin(5x_i) + c\varpi(x_i)$ where $x_i \in I = [0, 4]$ for $i = 0, \cdots, 1000 = M$, and $c = 0.25$ is adjusted in such a way that the signal-to-noise ratio $SNR = 10\log_{10}\left(\frac{\sum |y^{\delta}(x_i)|^2}{\sum |c\varpi(x_i)|^2}\right)$ is equal to $SNR = 10$dB. The noisy signal is shown in Figure 1.

Firstly, we estimate the half order derivative of y. For the DFOSGD, we take $N = 14$ and $\theta = 5$ which are the same values as the ones used in [3]. For the fractional Jacobi differentiator, we take $N = 14$ and $\kappa = \mu = 0$. The associated absolute estimations errors in the noise-free case and in the noisy case are given in Figure 2 and Figure 3 respectively, where the black dash-dotted lines represent the errors obtained by the fractional Jacobi differentiator and the red dotted lines represent the ones obtained by the DFOSGD. Moreover, we can see in Figure 4 the associated absolute noise error contributions. Secondly, we estimate the derivative $y^{(\alpha)}$ with $\alpha = 1.5$ by using the DFOSGD and the fractional Jacobi differentiator with $N = 14$, $\theta = 5$ and $\kappa = \mu = 0$. The obtained estimation errors are shown in Figure 5, Figure 6 and Figure 7. Consequently, we can observe that the fractional Jacobi differentiator is more accurate and more robust with noises than the DFOSGD in the noise-free case and in the noisy case.

IV. CONCLUSION

In this paper, we propose a fractional Jacobi differentiator which is a fractional order differentiator exactly given by an integral formula. This differentiator is deduced from the Jacobi orthogonal polynomial filter and the Riemann-Liouville fractional order derivative definition. It can accurately and easily estimate the fractional order derivatives of noisy signals. There are some parameters h, μ, κ and N on which the fractional Jacobi differentiator depends. We do not give any analysis on the influence of these parameters on the estimation errors. However, similarly to the integer order differentiation

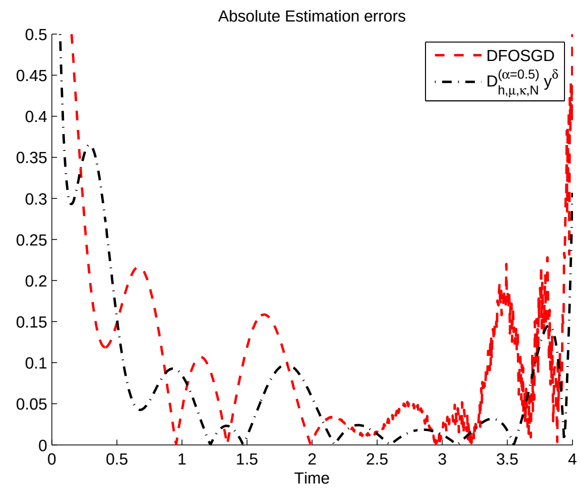

Fig. 3. Absolut estimation errors in noisy case.

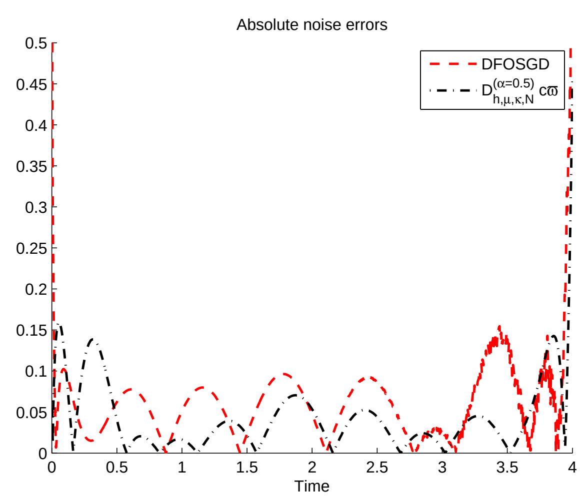

Fig. 4. Absolut noise error contributions.

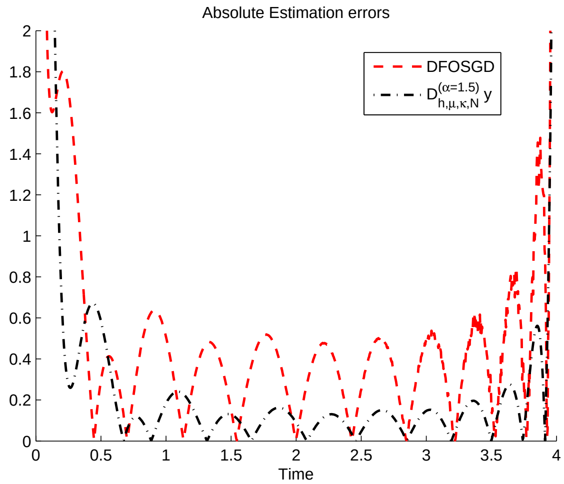

Fig. 5. Absolut estimation errors in noise-free case.

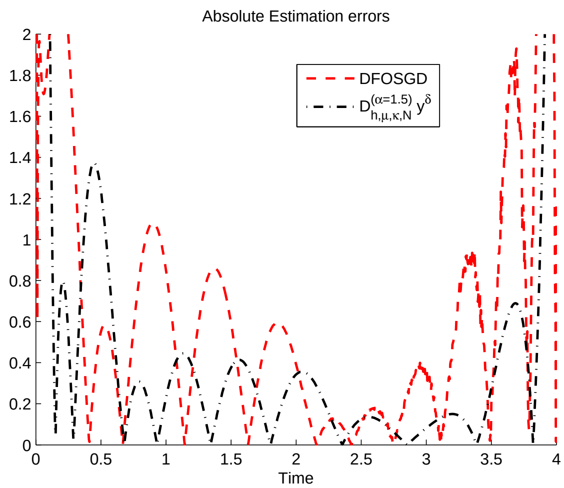

Fig. 6. Absolut estimation errors in noisy case.

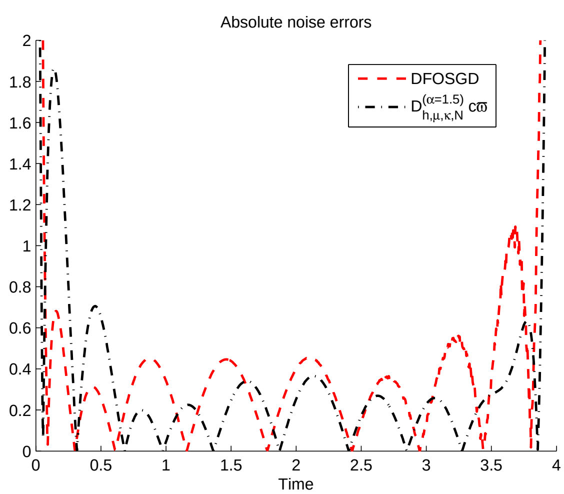

Fig. 7. Absolut noise error contributions.

by integration case (see [16], [17], [15], [14]), this study can be easily carried out in a further work. Moreover, we do not use the causal propriety of the fractional Jacobi differentiator in the numerical simulations. It will be used in some interesting applications where the sampling data may be irregularly spaced.

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